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Short Communication

Imaginary, sink and source flows in the vicinity of the separatrix of non-smooth dynamic systems

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Abstract

In this letter, the real and imaginary flows for non-smooth dynamical systems are described, and the δ-layer of the separation boundary is introduced as well. The onset, existence and vanishing of the sink and source flows in the δ-layer are presented. The switching between the two semi-passable flows and the switching between the sink and source flows are investigated through the singular gluing points. Finally, the necessary and sufficient conditions for the onset, vanishing and switching are presented. These conditions provide the criteria to determine the sliding motion on the separatrix, which can be very easily applied to non-smooth dynamical systems.

1. Introduction

In 2005, Luo [1] presented a local theory for non-smooth dynamical systems on the connectable and accessible domains. In this letter, the onset, existence and disappearance of the sink and source flows on the separatrix will be investigated, and the imaginary flows in non-smooth dynamical systems will be introduced to obtain the appropriate criteria for the onset of the sink and source flows. Before discussion, the accessible and inaccessible domains in phase space are first introduced herein. The accessible domain is a domain on which a specific dynamical system

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can be defined. However, the inaccessible domain is a domain on which no dynamical systems can be defined. Consider a planar, dynamic system consisting of n sub-dynamic systems in a universal domain $\Omega \subset \Re^2$. The universal domain is divided into n accessible sub-domains Ω_i, and the union of all the accessible sub-domains $\bigcup_{i=1}^{n} \Omega_i$ and the universal domain $\Omega = \bigcup_{i=1}^{n} \Omega_i \cup \Xi$, as shown in Fig. 1. Ξ is the union of the inaccessible domains. For the connectable and accessible domain in Fig.1, $\Xi = \{\emptyset\}$.

Definition 1. The C^{r+1}-continuous flow $\mathbf{x}_i^{(i)}(t) = \mathbf{\Phi}^{(i)}(\mathbf{x}_i^{(i)}(t_0), t, \boldsymbol{\mu}_i)$ is a real flow in the ith open sub-domain Ω_i, which is determined by a C^r-continuous system ($r \geqslant 1$) on Ω_i in a form of

$$\dot{\mathbf{x}}_i^{(i)} \equiv \mathbf{F}^{(i)}(\mathbf{x}_i^{(i)}, t, \boldsymbol{\mu}_i) \in \Re^2, \quad \mathbf{x}_i^{(i)} = (x_i^{(i)}, y_i^{(i)})^{\mathrm{T}} \in \Omega_i, \tag{1}$$

with the initial condition

$$\mathbf{x}_i^{(i)}(t_0) = \mathbf{\Phi}^{(i)}(\mathbf{x}_i^{(i)}(t_0), t_0, \boldsymbol{\mu}_i). \tag{2}$$

Notice that time is t and $\dot{\mathbf{x}}_i^{(i)} = \mathrm{d}\mathbf{x}_i^{(i)}/\mathrm{d}t$. In the sub-domain Ω_i, the vector field $\mathbf{F}^{(i)}(\mathbf{x}_i^{(i)}, t, \boldsymbol{\mu}_i) \equiv \mathbf{F}_i^{(i)}(t)$ with parameter vectors $\boldsymbol{\mu}_i = (\mu_{i1}, \mu_{i2}, \ldots, \mu_{in})^{\mathrm{T}} \in \Re^n$ is C^r-continuous ($r \geqslant 1$) in $\mathbf{x}$ and for all time t. $\mathbf{\Phi}^{(i)}(\mathbf{x}_i^{(i)}(t), t_0, \boldsymbol{\mu}_i) \equiv \mathbf{\Phi}_i^{(i)}(t)$. $\mathbf{x}_i^{(i)}(t)$ denotes the flow in the ith sub-domain Ω_i, governed by a dynamical system defined on the ith sub-domain Ω_i. In Ref. [1], the theory for non-smooth dynamical systems in Eq. (1) was developed from the following conditions:

A1: The switching between two adjacent sub-systems possesses time-continuity.
A2: For an unbounded, accessible sub-domain Ω_i, the corresponding vector field and its flow are bounded, i.e.,

$$\|\mathbf{F}_i^{(i)}\| \leqslant K_1(\text{const}) \text{ on } \Omega_i, \quad \text{and} \quad \|\mathbf{\Phi}_i^{(i)}\| \leqslant K_2(\text{const}) \text{ for } t \in [0, \infty) \tag{3}$$

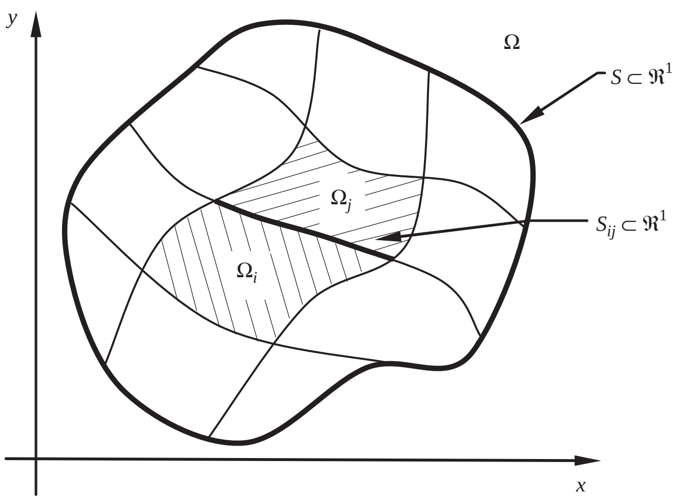

Fig. 1. A connectable, accessible domain in phase plane.

A3: For a bounded, accessible domain Ω_i, the corresponding vector field is bounded, but the flow may be unbounded, i.e.,

$$\|\mathbf{F}_i^{(i)}\| \leqslant K_1(\text{const}) \text{ on } \Omega_i, \quad \text{and} \quad \|\mathbf{\Phi}_i^{(i)}\| < \infty \text{ for } t \in [0, \infty). \tag{4}$$

Consider a boundary set of any two accessible sub-domains, formed by the intersection of the closed sub-domains, i.e., $\partial\Omega_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$ $(i, j \in \{1, 2, \ldots, n\},\ j \neq i)$, as shown in Fig. 2.

Definition 2. The boundary set in the 2-D phase space is defined as

$$S_{ij} \equiv \partial\Omega_{ij} = \{(x, y) | \forall (x, y) \in \bar{\Omega}_i \cap \bar{\Omega}_j \subset \Re^1 \text{ satisfying } H_{ij}(x, y) = 0\}. \tag{5}$$

In non-smooth dynamical systems, the normal vector of the boundary $\partial\Omega_{ij}$ is very important for investigating the local dynamics in the vicinity of the boundary. Therefore, from Eq. (5), we have

$$\mathbf{n}_{\partial\Omega_{ij}} = \nabla H_{ij} = \left(\frac{\partial H_{ij}}{\partial x}, \frac{\partial H_{ij}}{\partial y} \right)^{\mathrm{T}}_{(x_m, y_m)}. \tag{6}$$

To describe the dynamical characteristics of flows near the separation boundary in non-smooth dynamical systems, the δ-layer of the boundary $\partial\Omega_{ij}$ is introduced that is the neighborhood of $\partial\Omega_{ij}$ in phase plane, as shown in Fig. 3. Similarly, the δ-sub-layers in the neighborhood of $\partial\Omega_{ij}$ are described. The δ-sub-layer $\delta\Omega_\alpha (\alpha \in \{i, j\})$ are expressed by the shaded areas in Fig. 3. The following mathematical description of the δ-layer of the boundary $\partial\Omega_{ij}$ is given.

Definition 3. The δ-layer of the boundary $\partial\Omega_{ij}$ are defined as

$$\delta\Omega_{ij} \equiv \delta\Omega_i \cup \delta\Omega_j \cup \partial\Omega_{ij}, \tag{7}$$

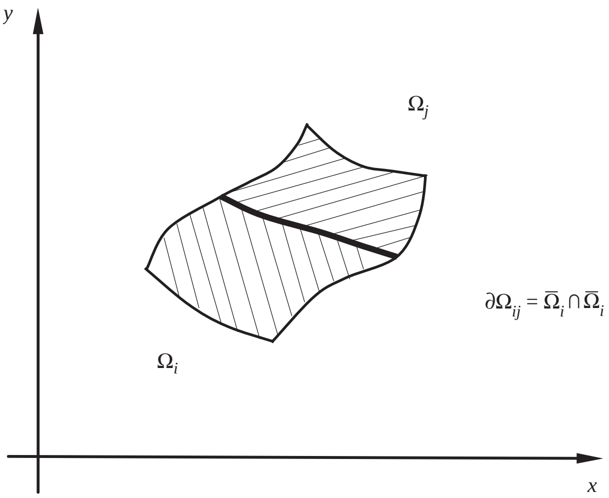

Fig. 2. Sub-domains Ω_i and Ω_j, the corresponding boundary $\partial\Omega_{ij}$.

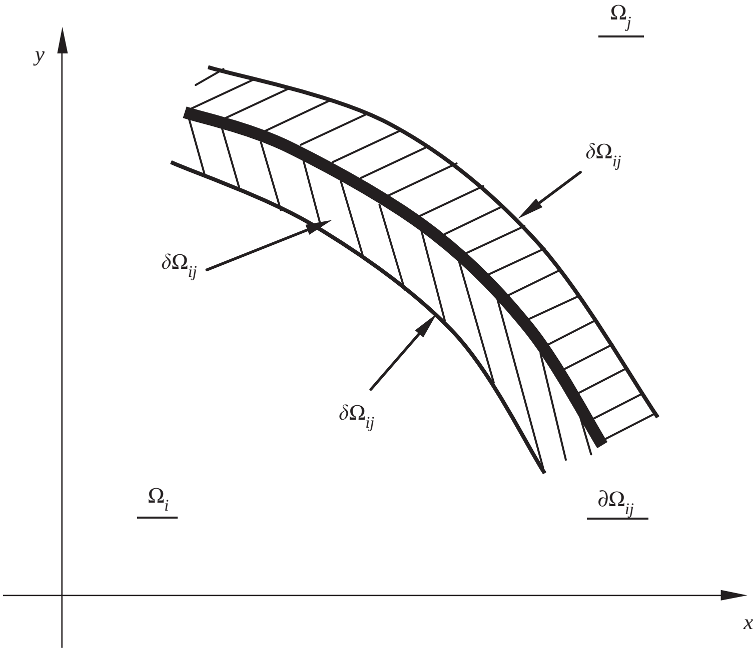

Fig. 3. The δ-sub-layers of boundary $\partial\Omega_{ij}$.

with the δ-sub-layer in Ω_α $\alpha \in \{i, j\}$ defined as

$$\delta\Omega_\alpha = \{\mathbf{x} \in \Omega_\alpha | \forall \delta_\alpha > 0, \|\mathbf{x} - \mathbf{x}^{(0)}\| < \delta_\alpha, \mathbf{x}^{(0)} \in \partial\Omega_{ij}\}. \tag{8}$$

For forming a separation boundary, besides for the semi- and non-passable boundaries, the gluing point connecting two portions of separatrix on the boundary is very important, and on the new boundary the flows possess two different vector characteristics. As in Ref. [1], the gluing points are defined as follows.

Definition 4. A countable point set on the boundary $\partial\Omega_{ij}$,

$$\Gamma_{ij} = \left\{ \mathbf{x}_{(k)}^{(0)} \in \partial\Omega_{ij} | \mathbf{x}_\alpha^{(\alpha)} \in \Omega_\alpha, \lim_{\mathbf{x}_\alpha^{(\alpha)} \to \mathbf{x}_{(k)}^{(0)}} (\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}) = 0, \alpha \in \{i, j\}, k \in \mathbb{N} \right\}, \tag{9}$$

is termed the gluing point set, and $\mathbb{N}$ is the natural number set.

Definition 5. A countable point set on the boundary $\partial\Omega_{ij}$,

$$\Gamma_{ij}^{(\alpha)} = \left\{ \mathbf{x}_{(k)}^{(0)} \in \partial\Omega_{ij} | \mathbf{x}_\alpha^{(\alpha)} \in \Omega_\alpha, \lim_{\mathbf{x}_\alpha^{(\alpha)} \to \mathbf{x}_{(k)}^{(0)}} (\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}) = 0, \alpha = \{i \text{ or } j\}, k \in \mathbb{N} \right\} \subset \Gamma_{ij}, \tag{10}$$

is termed the input or output, semi-gluing, singular points sets on the boundary.

The above definition $\Gamma_{ij}^{(\alpha)}$ indicates the switching of the flow direction at the singular point on the side of Ω_α.

Definition 6. A countable point set on the boundary $\partial\Omega_{ij}$,

$$\Gamma_{ij}^{(0)} = \left\{ \mathbf{x}_k^{(0)} \in \partial\Omega_{ij} | \mathbf{x}_\alpha^{(\alpha)} \in \Omega_\alpha, \lim_{\mathbf{x}_\alpha^{(\alpha)} \to \mathbf{x}_k^{(0)}} (\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}) = 0, \alpha = \{i \text{ and } j\}, k \in \mathbb{N} \right\} \subset \Gamma_{ij}, \tag{11}$$

is termed the full-gluing, singular point set.

The foregoing definition $\Gamma_{ij}^{(0)}$ indicates the switching of the flow direction at the singular point on both sides of Ω_α. The gluing point set is $\Gamma_{ij} = \Gamma_{ij}^{(i)} \cup \Gamma_{ij}^{(j)} \cup \Gamma_{ij}^{(0)}$. The gluing points connecting the semi- and non-passable boundaries will form a new separation boundary. The definitions and theorems for semi- and non-passable boundaries were presented in Ref. [1], which are used in the further discussion. Therefore, they are briefly stated as follows.

Definition 7. For a discontinuous dynamical system in Eq. (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at t_m. $\forall\ \varepsilon > 0$, $\exists\ [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}_i^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m+})$. The non-empty boundary set $\partial\Omega_{ij}$ to a flow $\mathbf{x}_\alpha^{(\alpha)}(t)$ ($\alpha \in \{i,j\}$) is semi-passable from the domain Ω_i to Ω_j (expressed by $\overrightarrow{\partial\Omega}_{ij}$) if the flow $\mathbf{x}_\alpha^{(\alpha)}(t)$ possesses the following properties:

$$\begin{aligned} &\text{either} \quad \left\{ \begin{array}{l} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet [\mathbf{x}_i^{(i)}(t_{m-}) - \mathbf{x}_i^{(i)}(t_{m-\varepsilon})] > 0 \text{ and} \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet [\mathbf{x}_j^{(j)}(t_{m+\varepsilon}) - \mathbf{x}_j^{(j)}(t_{m+})] > 0 \end{array} \right\} \quad \text{for } \partial\Omega_{ij} \text{ convex to } \Omega_j, \\ &\text{or} \quad \left\{ \begin{array}{l} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet [\mathbf{x}_i^{(i)}(t_{m-}) - \mathbf{x}_i^{(i)}(t_{m-\varepsilon})] < 0 \text{ and} \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet [\mathbf{x}_j^{(j)}(t_{m+\varepsilon}) - \mathbf{x}_j^{(j)}(t_{m+})] < 0 \end{array} \right\} \quad \text{for } \partial\Omega_{ij} \text{ convex to } \Omega_i. \end{aligned} \tag{12}$$

Theorem 1. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists\ [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}_i^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m+})$ *and, both* $\mathbf{x}_i^{(i)}(t)$ *and* $\mathbf{x}_j^{(j)}(t)$ *are* $C_{[t_{m-\varepsilon}, t_m)}^r$ *and* $C_{(t_m, t_{m+\varepsilon}]}^r$*-continuous* ($r \geqslant 2$) *for time t, respectively, and* $\|\mathrm{d}^r \mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^r\| < \infty (\alpha \in \{i,j\})$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is semi-passable from the domain* Ω_i *to* Ω_j *if*

$$\begin{aligned} &\textit{either } \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{m-}) > 0 \textit{ and } \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{m+}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_j, \\ &\textit{or } \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{m-}) < 0 \textit{ and } \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{m+}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_i. \end{aligned} \tag{13}$$

Theorem 2. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\ \varepsilon > 0$, $\exists\ [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}_i^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}_j^{(j)}(t_{m+})$ *and, both* $\mathbf{F}_i^{(i)}(t)$ *and* $\mathbf{F}_j^{(j)}(t)$ *are* $C_{[t_{m-\varepsilon}, t_m)}^r$ *and* $C_{(t_m, t_{m+\varepsilon}]}^r$*-continuous* ($r \geqslant 1$) *for time t, respectively and* $\|\mathrm{d}^{r+1} \mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty (\alpha \in \{i,j\})$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is semi-passable from the domain* Ω_i *to* Ω_j *iff*

$$\begin{aligned} &\textit{either } \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \mathbf{F}_i^{(i)}(t_{m-}) > 0 \textit{ and } \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \mathbf{F}_j^{(j)}(t_{m+}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_j, \\ &\textit{or } \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \mathbf{F}_i^{(i)}(t_{m-}) < 0 \textit{ and } \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \mathbf{F}_j^{(j)}(t_{m+}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_i, \end{aligned} \tag{14}$$

where $\mathbf{F}_i^{(i)}(t_{m-}) = \mathbf{F}^{(i)}(\mathbf{x}_i^{(i)}, t_{m-}, \boldsymbol{\mu}_i)$ *and* $\mathbf{F}_j^{(j)}(t_{m+}) = \mathbf{F}^{(j)}(\mathbf{x}_j^{(j)}, t_{m+}, \boldsymbol{\mu}_j)$.

Definition 8. For a discontinuous dynamical system in Eq. (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists\ [t_{m-\varepsilon}, t_m)$, suppose $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ for $\alpha \in \{i,j\}$, so that the non-empty boundary set $\partial\Omega_{ij}$ is the non-passable boundary of the first kind, $\widetilde{\partial\Omega_{ij}}$ (or termed a sink boundary between the sub-domains Ω_i and Ω_j) if the flows $\mathbf{x}_\gamma^{(\gamma)}(t)$ for ($\gamma \in \{\alpha, \beta\} \in \{i,j\}$ and $\alpha \neq \beta$) in the neighborhood of the boundary $\partial\Omega_{ij}$ possess the following properties:

$$\{\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet [\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) - \mathbf{x}_\alpha^{(\alpha)}(t_{m-\varepsilon})]\} \times \{\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet [\mathbf{x}_\beta^{(\beta)}(t_{m-}) - \mathbf{x}_\beta^{(\beta)}(t_{m-\varepsilon})]\} < 0. \tag{15}$$

Definition 9. For a discontinuous dynamical system in Eq. (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists\ (t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}_\alpha^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ for $\alpha \in \{i,j\}$, so that the non-empty boundary set $\partial\Omega_{ij}$ is the non-passable boundary of the second kind $\widehat{\partial\Omega_{ij}}$ (or termed a source boundary between the sub-domains Ω_i and Ω_j) if the flows $\mathbf{x}_\gamma^{(\gamma)}(t)$ for ($\gamma \in \{\alpha, \beta\} \in \{i,j\}$ and $\alpha \neq \beta$) in the neighborhood of the boundary $\partial\Omega_{ij}$ possess the following properties:

$$\{\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet [\mathbf{x}_\alpha^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}_\alpha^{(\alpha)}(t_{m+})]\} \times \{\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet [\mathbf{x}_\beta^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}_\beta^{(\beta)}(t_{m+})]\} < 0. \tag{16}$$

Theorem 3. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists\ [t_{m-\varepsilon}, t_m)$, *suppose* $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ ($\alpha \in \{i,j\}$) *and,* $\mathbf{x}_\alpha^{(\alpha)}$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$*-continuous* ($r \geqslant 2$) *for time t and* $\|\mathrm{d}^r \mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is a non-passable boundary of the first kind if*

$$[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{m-})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_\beta^{(\beta)}(t_{m-})] < 0 \tag{17}$$

for $\{\alpha, \beta\} \in \{i,j\} (\alpha \neq \beta)$.

Theorem 4. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists\ [t_{m-\varepsilon}, t_m)$, *suppose* $\mathbf{x}_\alpha^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ ($\alpha \in \{i,j\}$) *and,* $\mathbf{F}_\alpha^{(\alpha)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$*-continuous* ($r \geqslant 1$) *and* $\|\mathrm{d}^{r+1} \mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is a non-passable boundary of the first kind if for* $\beta \in \{i,j\}$ ($\alpha \neq \beta$),

$$[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \mathbf{F}_\alpha^{(\alpha)}(t_{m-})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \mathbf{F}_\beta^{(\beta)}(t_{m-})] < 0, \tag{18}$$

where $\mathbf{F}_\alpha^{(\alpha)}(t_{m-}) \triangleq \mathbf{F}^{(\alpha)}(\mathbf{x}_\alpha^{(\alpha)}, t_{m-}, \boldsymbol{\mu}_\alpha)$ *and* $\mathbf{F}_\beta^{(\beta)}(t_{m-}) \triangleq \mathbf{F}^{(\beta)}(\mathbf{x}_\beta^{(\beta)}, t_{m-}, \boldsymbol{\mu}_\beta)$.

Theorem 5. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists\ (t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}_\alpha^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ ($\alpha \in \{i,j\}$) *and* $\mathbf{x}_\alpha^{(\alpha)}$ *is* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 2$) *for time t and* $\|\mathrm{d}^r \mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is a non-passable boundary of the second kind if*

$$[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{m+})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_\beta^{(\beta)}(t_{m+})] < 0 \tag{19}$$

for $\beta \in \{i,j\}$ ($\alpha \neq \beta$).

Theorem 6. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists\ (t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ ($\alpha \in \{i,j\}$) *and* $\mathbf{F}_\alpha^{(\alpha)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$*-continuous* ($r \geqslant 1$) *and* $\|\mathrm{d}^{r+1} \mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is a non-passable boundary of the second* *kind if for* $\beta \in \{i,j\}$ $(\alpha \neq \beta)$,

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\alpha}(t_{m+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\beta}(t_{m+})] < 0, \tag{20}$$

where $\mathbf{F}^{(\alpha)}_{\alpha}(t_{m+}) \triangleq \mathbf{F}^{(\alpha)}_{\alpha}(\mathbf{x}, t_{m+}, \mu_\alpha)$ *and* $\mathbf{F}^{(\beta)}_{\beta}(t_{m+}) \triangleq \mathbf{F}^{(\beta)}(\mathbf{x}, t_{m+}, \boldsymbol{\mu}_\beta)$.

2. Main results

The above definitions and theorems of semi- and non-passable boundaries are based on the real flows of the non-smooth dynamical system in Eq. (1). The real flow $\mathbf{x}^{(i)}_i(t)$ in Ω_i is governed by a dynamical system on its own domain. However, another flow $\mathbf{x}^{(j)}_i$ in Ω_i is governed by a dynamical system defined on the jth sub-domain Ω_j, which is of great interest in this letter. This kind of flow is called the *imaginary flow* because the flow is not determined by the dynamical system on its own domain. To further understand the dynamical behavior of the non-smooth dynamical system, it is necessary to introduce the *imaginary flows*. Consider the jth imaginary flow in the ith-domain Ω_i is a flow in Ω_i governed by the dynamical system defined on the jth-sub-domain Ω_j. The flow is not a real one governed by the non-smooth dynamical system, thus this flow is termed the *imaginary flow* in this sense. In additions, the two sub-domains can be either adjacent or separable. The mathematical definition of imaginary flows is as follows.

Definition 10. The C^{r+1} $(r \geqslant 1)$-continuous flow $\mathbf{x}^{(j)}_i(t)$ is termed the jth-imaginary flow in the ith open sub-domain Ω_i if the flow $\mathbf{x}^{(j)}_i(t)$ is determined by application of a C^r-continuous system, defined on the jth open sub-domain Ω_j, to the ith open sub-domain Ω_i, i.e.,

$$\dot{\mathbf{x}}^{(j)}_i \equiv \mathbf{F}^{(j)}(\mathbf{x}^{(j)}_i, t, \boldsymbol{\mu}_j) \in \Re^2, \quad \mathbf{x}^{(j)}_i = (x^{(j)}_i, y^{(j)}_i)^{\mathrm{T}} \in \Omega_i, \tag{21}$$

with the initial conditions

$$\mathbf{x}^{(j)}_i(t_0) = \boldsymbol{\Phi}^{(j)}(\mathbf{x}^{(j)}_i(t_0), t_0, \boldsymbol{\mu}_j). \tag{22}$$

To demonstrate the above concept, the real and imaginary flows in two adjacent sub-domains are illustrated in Fig. 4 for the semi-passable boundary, the sink boundary and the source boundary. The boundary point $\mathbf{x}_m$ is at time t_m, and $\mathbf{x}^{(\alpha)}_\alpha(t_m) = \mathbf{x}_m = \mathbf{x}^{(\beta)}_\alpha(t_m)$ for $\{\alpha, \beta\} = \{i,j\}$ and $\alpha \neq \beta$. The real flows $\mathbf{x}^{(\alpha)}_\alpha(t)$ are represented by solid curves. The imaginary flows $\mathbf{x}^{(\beta)}_\alpha(t)$ in the domain Ω_α are depicted by dashed curves. $\mathbf{x}^{(\alpha)}_\alpha(t_{m\pm\varepsilon})$ and $\mathbf{x}^{(\beta)}_\alpha(t_{m\pm\varepsilon})$ in the δ-layer are the values of the *real* and *imaginary* flows at $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ for an arbitrary $\varepsilon > 0$. As $\varepsilon \to 0$, $\{\mathbf{x}^{(\alpha)}_\alpha(t_{m\pm\varepsilon}), \mathbf{x}^{(\beta)}_\alpha(t_{m\pm\varepsilon})\} \to \mathbf{x}_m$. From the foregoing definition, the flow $\mathbf{x}^{(\alpha)}_\alpha(t) \cup \mathbf{x}^{(\beta)}_\alpha(t)$ gives a continuous flow in the two sub-domains Ω_i and Ω_j plus the boundary $\partial\Omega_{ij}$. Therefore, Definitions 7–9 for the real flows near the semi-passable, sink and source boundaries are applicable to the imaginary flows. Similarly, Theorems 1–6 are applicable to the imaginary flows.

Theorem 7. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists$ $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$ *in the* δ*-layer of* $\partial\Omega_{ij}$. *Suppose* $\{\mathbf{x}^{(\alpha)}_\alpha(t_{m-}), \mathbf{x}^{(\beta)}_\beta(t_{m+})\} = \mathbf{x}_m$ *and*

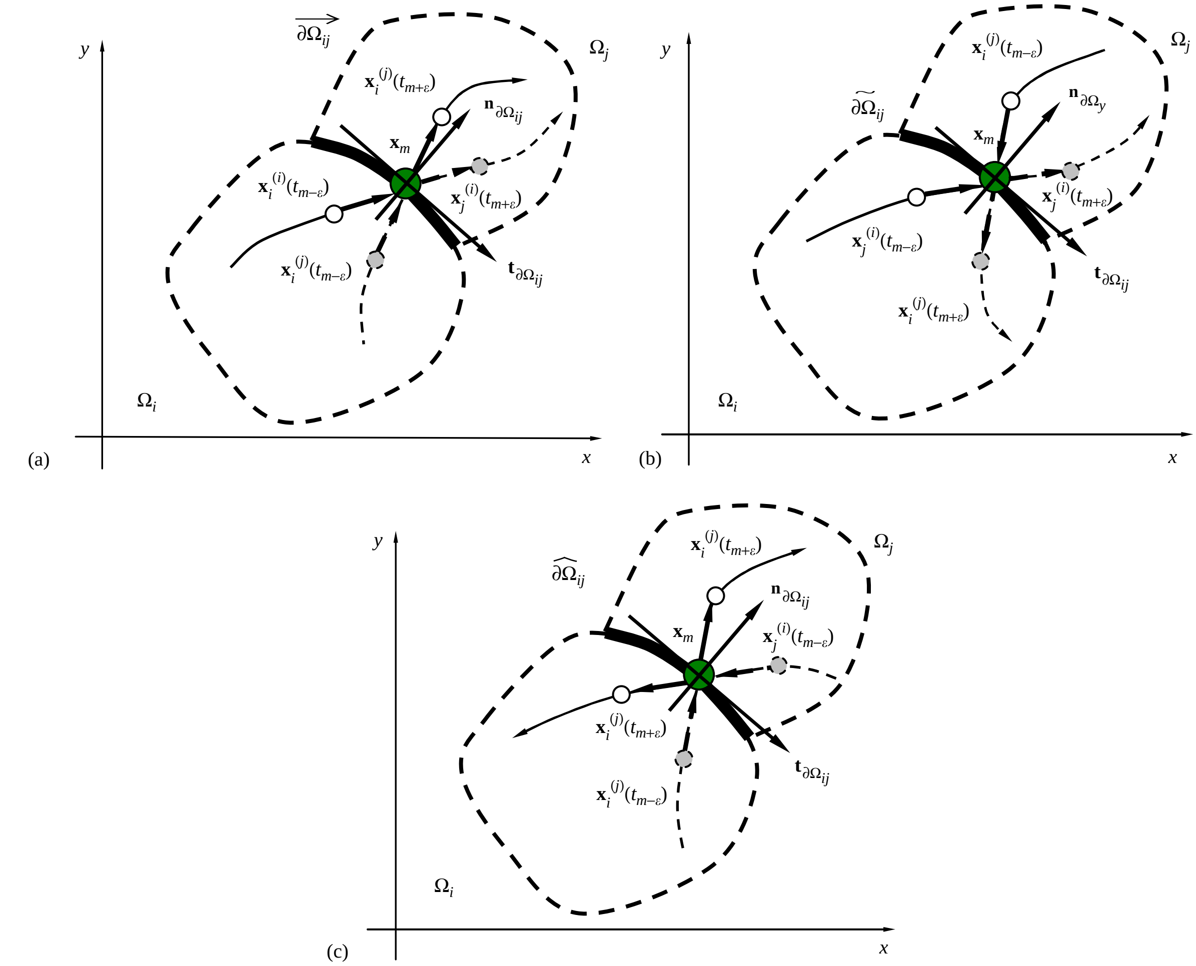

Fig. 4. Real and imaginary flows near: (a) the semi-passable boundary, (b) the sink boundary and (c) the source boundary. The boundary point $\mathbf{x}_m$ is at time t_m. The real flows $\mathbf{x}_\alpha^{(\alpha)}(t)$ for $\alpha = \{i, j\}$ are represented by the solid curves. The imaginary flows $\mathbf{x}_\alpha^{(\beta)}(t)$ in the domain Ω_α for $\beta = \{i, j\}$ and $\alpha \neq \beta$ are depicted by the dashed curves. Two vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ are the normal and tangential vectors of the boundary curve $\partial\Omega_{ij}$ determined by $H_{ij}(x, y) = 0$. $\mathbf{x}_\alpha^{(\alpha)}(t_{m\pm\varepsilon})$ and $\mathbf{x}_\alpha^{(\beta)}(t_{m\pm\varepsilon})$ are the values of the real and imaginary flows at $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ for an arbitrary $\varepsilon > 0$.

$\{\mathbf{x}_\alpha^{(\beta)}(t_{m-}), \mathbf{x}_\beta^{(\alpha)}(t_{m+})\} = \mathbf{x}_m$ *(*$\{\alpha, \beta\} \in \{i, j\}$ *and* $\alpha \neq \beta$*) hold, the real and imaginary flows* $\{\mathbf{x}_\alpha^{(\alpha)}(t), \mathbf{x}_\alpha^{(\beta)}(t)\}$ *and* $\{\mathbf{x}_\beta^{(\alpha)}(t), \mathbf{x}_\beta^{(\beta)}(t)\}$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time* t*, respectively.* $\{\|\mathrm{d}^r\mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^r\|, \|\mathrm{d}^r\mathbf{x}_\alpha^{(\beta)}/\mathrm{d}t^r\|\} < \infty$*. The boundary set* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *to the real and imaginary flows is semi-passable iff*

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{m-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\alpha^{(\beta)}(t_{m-})] > 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\beta^{(\alpha)}(t_{m+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\alpha^{(\beta)}(t_{m+})] > 0. \tag{23}$$

Proof. Consider all points $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ in the δ-layer, and application of Theorem 1 to the real and imaginary flows gives the necessary and sufficient conditions as

$$\text{either } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \bullet \dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m-}) > 0 \text{ and } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \bullet \dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m+}) > 0 \quad \text{for } \partial\Omega_{\alpha\beta} \text{ convex to } \Omega_{\beta},$$
$$\text{or } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \bullet \dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m-}) < 0 \text{ and } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \bullet \dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m+}) < 0 \quad \text{for } \partial\Omega_{\alpha\beta} \text{ convex to } \Omega_{\alpha}.$$

and

$$\text{either } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \bullet \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m-}) > 0 \text{ and } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \bullet \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m+}) > 0 \quad \text{for } \partial\Omega_{\alpha\beta} \text{ convex to } \Omega_{\beta},$$
$$\text{or } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \bullet \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m-}) < 0 \text{ and } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \bullet \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m+}) < 0 \quad \text{for } \partial\Omega_{\alpha\beta} \text{ convex to } \Omega_{\alpha}.$$

Therefore, no matter how $\partial\Omega_{\alpha\beta}$ is convex to either Ω_α or Ω_β, the real flows require

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m+})] > 0$$

and the imaginary flows require

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m-})] > 0$$

Because the real and imaginary flows $\{\mathbf{x}^{(i)}_i(t), \mathbf{x}^{(i)}_j(t)\}$ and $\{\mathbf{x}^{(j)}_i(t), \mathbf{x}^{(j)}_j(t)\}$, respectively, are $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r \geqslant 2$) for time t, the following relations at $t = t_{m\pm}$ hold:

$$\left.\frac{d^s\mathbf{x}^{(\alpha)}_{\alpha}}{\mathrm{d}t^s}\right|_{t=t_{m-}} = \left.\frac{d^s\mathbf{x}^{(\alpha)}_{\beta}}{\mathrm{d}t^s}\right|_{t=t_{m+}} \quad \text{and} \quad \left.\frac{d^s\mathbf{x}^{(\beta)}_{\alpha}}{\mathrm{d}t^s}\right|_{t=t_{m-}} = \left.\frac{d^s\mathbf{x}^{(\beta)}_{\beta}}{\mathrm{d}t^s}\right|_{t=t_{m+}}$$

for $s \in \{0, 1, \ldots, r\}$ and $\{\alpha, \beta\} \in \{i, j\}$ with $\alpha \neq \beta$. From the foregoing equation, the following equations hold:

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(\alpha)}_{\alpha}(t_{m-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(\beta)}_{\alpha}(t_{m-})] > 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{m+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{m+})] > 0$$

which implies that the boundary set $\overrightarrow{\partial\Omega}_{\alpha\beta}$ to the real and imaginary flows is semi-passable and vice versa. This theorem is proved. □

Theorem 8. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists\, [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$ *in the* δ*-layer of* $\partial\Omega_{ij}$. *Suppose* $\{\mathbf{x}^{(\alpha)}_{\alpha}(t_{m-}), \mathbf{x}^{(\beta)}_{\beta}(t_{m+})\} = \mathbf{x}_m$ *and* $\{\mathbf{x}^{(\beta)}_{\alpha}(t_{m-}), \mathbf{x}^{(\alpha)}_{\beta}(t_{m+})\} = \mathbf{x}_m$ ($\{\alpha, \beta\} \in \{i, j\}$ *and* $\alpha \neq \beta$) *hold, the vector fields* $\{\mathbf{F}^{(\alpha)}_{\alpha}(t), \mathbf{F}^{(\alpha)}_{\beta}(t)\}$ *and* $\{\mathbf{F}^{(\beta)}_{\alpha}(t), \mathbf{F}^{(\beta)}_{\beta}(t)\}$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 1$) *for time* t, *respectively.* $\{\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}_{\alpha}/\mathrm{d}t^r\|, \|\mathrm{d}^{r+1}\mathbf{x}^{(\beta)}_{\alpha}/\mathrm{d}t^{r+1}\|\} < \infty$. *The boundary set* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *to the real and imaginary flows is semi-passable if*

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\alpha}(t_{m-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\alpha}(t_{m-})] > 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\beta}(t_{m+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\beta}(t_{m+})] > 0. \tag{24}$$

Proof. Application of Definitions 1 and 9 to Theorem 8 generated Eq. (24). This theorem is proved. □

Lemma 1. *For a discontinuous system in Eq.* (1), *a point* $\mathbf{x}^{(0)} \in \partial\Omega_{ij}$ *is* (i) *a gluing point* $\mathbf{x}^{(0)} \in \Gamma_{ij} \equiv \bigcup_\alpha \Gamma_{ij}^{(\alpha)} \bigcup \Gamma_{ij}^{(0)}$ *for* $\alpha = \{i, j\}$, *or* (ii) *a semi-gluing point* $\mathbf{x}^{(0)} \in \Gamma_{ij}^{(\alpha)}$ *for* $\alpha = \{i \text{ or } j\}$ *or* (iii) *a full-gluing point* $\mathbf{x}^{(0)} \in \Gamma_{ij}^{(0)}$ *for* $\alpha = \{i \text{ and } j\}$ *if in the* δ*-layer of* $\partial\Omega_{ij}$,

$$\lim_{\mathbf{x}_\alpha^{(\alpha)}(t_{m\pm}) \to \mathbf{x}_{(k)}^{(0)}} (\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \mathbf{F}_\alpha^{(\alpha)}(t_{m\pm})) = 0, \qquad \lim_{\mathbf{x}_\alpha^{(\alpha)}(t_{m\pm}) \to \mathbf{x}_{(k)}^{(0)}} (\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \mathbf{F}_\alpha^{(\beta)}(t_{m\mp})) = 0. \tag{25}$$

Proof. Application of Definitions 1 and 10 to Definition 3–5 yields Eq. (25). This lemma is proved. □

In Ref. [1], the formation of discontinuous boundaries was discussed. However, the necessary and sufficient conditions for four basic formations of boundaries in planar, non-smooth, dynamical systems need to be developed. From the definitions of the semi-, non-passable boundaries and gluing points, the following theorems are presented for such necessary and sufficient conditions.

Theorem 9. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists\, [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$ *in the* δ*-layer of* $\partial\Omega_{ij}$. *Suppose* $\{\mathbf{x}_i^{(i)}(t_{m-}), \mathbf{x}_j^{(j)}(t_{m+})\} = \mathbf{x}_m$ *and* $\{\mathbf{x}_i^{(j)}(t_{m-}), \mathbf{x}_j^{(i)}(t_{m+})\} = \mathbf{x}_m$ *hold, the flows* $\{\mathbf{x}_i^{(i)}(t), \mathbf{x}_i^{(j)}(t)\}$ *and* $\{\mathbf{x}_j^{(i)}(t), \mathbf{x}_j^{(j)}(t)\}$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 2$) *for time t, respectively.* $\|\mathrm{d}^r \mathbf{x}_\alpha^{(\alpha)} / \mathrm{d}t^r\| < \infty$ ($\alpha \in \{i, j\}$) *and* $m \in \{p, q, n\}$.

(1) *A passable boundary* ($\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \overleftarrow{\partial\Omega}_{ij}$) *exists if*

$$\begin{aligned}
&[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{\lambda\mp})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_i^{(j)}(t_{\lambda\mp})] > 0,\\
&[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_j^{(i)}(t_{\lambda\pm})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{\lambda\pm})] > 0,\\
&[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{n\pm})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{n\mp})] = 0
\end{aligned} \tag{26}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{ij}$ *and* $\mathbf{x}_q \in \overleftarrow{\partial\Omega}_{ij}$ *and* $\lambda \in \{p, q\} \neq n$.

(2) *A non-passable boundary* ($\partial\Omega_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$) *exists if*

$$\begin{aligned}
&[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{p-})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{p-})] < 0,\\
&[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{q+})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{q+})] < 0,\\
&[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{n\pm})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{n\pm})] = 0
\end{aligned} \tag{27}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \widetilde{\partial\Omega}_{ij}$ *and* $\mathbf{x}_q \in \widehat{\partial\Omega}_{ij}$.

(3) *A mixed boundary of the first kind* ($\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widetilde{\partial\Omega}_{ij}$) *exists if*

$$\begin{aligned}
&[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{p-})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_\alpha^{(\beta)}(t_{p-})] > 0,\\
&[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{q-})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{q-})] < 0,\\
&[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{n-})] \times [\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{n-})] = 0
\end{aligned} \tag{28}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ *and* $\mathbf{x}_q \in \widetilde{\partial\Omega}_{ij}$.

(4) *A mixed boundary of the second kind* $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ *if*

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(\alpha)}_{\beta}(t_{p+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(\beta)}_{\beta}(t_{p+})] > 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(i)}_{i}(t_{q+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(j)}_{j}(t_{q+})] < 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(i)}_{i}(t_{n+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}^{(j)}_{j}(t_{n+})] = 0 \tag{29}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ *and* $\mathbf{x}_q \in \widehat{\partial\Omega}_{ij}$.

Proof. Using Theorem 7 and Lemma 1, this theorem is directly proved. □

Theorem 10. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists\ [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$ *in the* δ*-layer of* $\partial\Omega_{ij}$. *Suppose* $\{\mathbf{x}^{(i)}_i(t_{m-}), \mathbf{x}^{(j)}_j(t_{m+})\} = \mathbf{x}_m$ *holds, the vector fields* $\mathbf{F}^{(i)}_i(t)$ *and* $\mathbf{F}^{(j)}_j(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t, respectively.* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}_{\alpha}/\mathrm{d}t^{r+1}\| < \infty$ $(\alpha \in \{i, j\})$ *and* $m \in \{p, q, n\}$.

(1) *A passable boundary* ($\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \overleftarrow{\partial\Omega}_{ij}$) *exists if*

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{\lambda\mp})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{i}(t_{\lambda\mp})] > 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{j}(t_{\lambda\pm})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{\lambda\pm})] > 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{n\pm})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{n\pm})] = 0 \tag{30}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{ij}$ *and* $\mathbf{x}_q \in \overleftarrow{\partial\Omega}_{ij}$ *and* $\lambda \in \{p, q\} \neq n$.

(2) *A non-passable boundary* ($\partial\Omega_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$) *exists if*

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{p-})] < 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{q+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{q+})] < 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{n\pm})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{n\pm})] = 0 \tag{31}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \widetilde{\partial\Omega}_{ij}$ *and* $\mathbf{x}_q \in \widehat{\partial\Omega}_{ij}$.

(3) *A mixed boundary of the first kind* ($\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widetilde{\partial\Omega}_{ij}$) *exists if*

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\alpha}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\alpha}(t_{p-})] > 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{q-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{q-})] < 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{n-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{n-})] = 0 \tag{32}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ *and* $\mathbf{x}_q \in \widetilde{\partial\Omega}_{ij}$.

(4) *A mixed boundary of the second kind* $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ *if*

$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\beta}(t_{p+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\beta}(t_{p+})] > 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{q+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{q+})] < 0,$$
$$[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{n+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{n+})] = 0 \tag{33}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ *and* $\mathbf{x}_q \in \widehat{\partial\Omega}_{ij}$.

Proof. Application Definition 1 to Theorem 9 yields Eqs. (31)–(34), this theorem is proved. □

Using the definitions and theorems of the semi-passable, sink and source boundaries, the flow characteristics in the vicinity of the mixed boundary can be defined and the corresponding theorem can be developed. Based on this definition, the local nature of the flow near the mixed boundary can be analyzed through investigating the vector fields of the real and imaginary flows. The flow switching between the semi-passable flows or between non-passable flows are defined, and the onset, existence and disappearance of the sink and source flows can be obtained. The flow switching is described in the following definition.

Definition 11. For a discontinuous dynamical system in Eq. (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists\ [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$ in the δ-layer of $\partial\Omega_{ij}$. Suppose $\{\mathbf{x}_i^{(i)}(t_{m-}), \mathbf{x}_j^{(j)}(t_{m+})\} = \mathbf{x}_m$ holds, the flows $\mathbf{x}_i^{(i)}(t)$ and $\mathbf{x}_j^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon},t_m)}$ and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous ($r \geqslant 2$) for time t, respectively. $\|\mathrm{d}^r\mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^r\| < \infty$ ($\alpha \in \{i,j\}$) and $m \in \{p,q,n\}$.

(1) Two different semi-passable flows in the δ-layer of $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \overleftarrow{\partial\Omega}_{ij}$ are switched on the boundary if

$$
\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{n\pm}) = 0, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{n\pm}) = 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{q+})] < 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{p+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{q-})] < 0
\end{aligned} \tag{34}
$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{ij}$ and $\mathbf{x}_q \in \overleftarrow{\partial\Omega}_{ij}$ and $\lambda \in \{p,q\} \neq n$.

(2) The source and sink flows in the δ-layer of $\partial\Omega_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ are switched on the boundary if

$$
\begin{aligned}
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{n\pm})] = [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{n\pm})] = 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_i^{(i)}(t_{q+})] < 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_j^{(j)}(t_{q+})] < 0
\end{aligned} \tag{35}
$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \widetilde{\partial\Omega}_{ij}$ and $\mathbf{x}_q \in \widehat{\partial\Omega}_{ij}$.

(3) The sliding flow in the δ-layer of $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widetilde{\partial\Omega}_{ij}$ appears or disappears if

$$
\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\beta^{(\beta)}(t_{n-}) = 0, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{n-}) \neq 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{q-})] > 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\alpha^{(\beta)}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\beta^{(\beta)}(t_{q-})] < 0
\end{aligned} \tag{36}
$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ and $\mathbf{x}_q \in \widetilde{\partial\Omega}_{ij}$.

(4) The source flow in the δ-layer of $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ appears or disappears if

$$
\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{n+}) = 0, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\beta^{(\beta)}(t_{n+}) \neq 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\beta^{(\alpha)}(t_{p+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\alpha^{(\alpha)}(t_{q+})] < 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\beta^{(\beta)}(t_{p+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \dot{\mathbf{x}}_\beta^{(\beta)}(t_{q+})] > 0
\end{aligned} \tag{37}
$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ and $\mathbf{x}_q \in \widehat{\partial\Omega}_{ij}$.

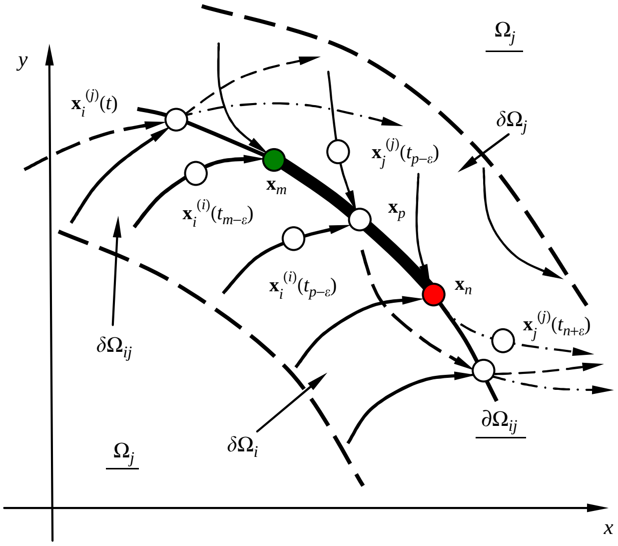

Fig. 5. The onset, existence and disappearance of the sliding flows in the δ-sub-layer of boundary $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \partial\Omega_{ij} \cup \overrightarrow{\partial\Omega}_{ij}$. The solid, and dash–dotted curves represent $\mathbf{x}_\alpha^{(\alpha)}(t)$ $\alpha \in \{i,j\}$ for $(t \in [t_{m-\varepsilon}, t_{m-}))$ and $(t \in (t_{m+}, t_{m+\varepsilon}])$ accordingly. The dash curves represent $\mathbf{x}_\alpha^{(\beta)}(t)$ $\alpha \neq \beta \in \{i,j\}$. The dark and light curves represent the flows in Ω_i and Ω_j, respectively. The points $\mathbf{x}_m$ and $\mathbf{x}_n$ are the appearance and disappearance of the sliding motions. $\mathbf{x}_p$ is all the points on the sliding boundary.

From the above definition, the onset, existence and disappearance of the sliding flows on the boundary $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \widetilde{\partial\Omega}_{ij} \cup \overrightarrow{\partial\Omega}_{ij}$ in the δ-sub-layer are intuitively illustrated in Fig. 5 through the real and imaginary flows. The solid and dash–dotted curves are used for $\mathbf{x}_\alpha^{(\alpha)}(t)$ $(\alpha \in \{i,j\})$ for $(t \in [t_{m-\varepsilon}, t_{m-}))$ and $(t \in (t_{m+}, t_{m+\varepsilon}])$ accordingly. The dash curves depict the imaginary flows $\mathbf{x}_\alpha^{(\beta)}(t)$ $(\alpha \neq \beta \in \{i,j\})$ in the two domains. The dark and light curves represent the flows in Ω_i and Ω_j, respectively. The points $\mathbf{x}_m$ and $\mathbf{x}_n$ are the appearance and disappearance of the sliding motions. $\mathbf{x}_p \in \widetilde{\partial\Omega}_{ij}$ is all the points on the sliding boundary. Similarly, the onset, existence and disappearance of the source flows of the boundary $\partial\Omega_{ij}$can be described. The switching between the two semi-passable boundaries and between the source and sink boundaries can be illustrated.

Theorem 11. *For a discontinuous dynamical system in Eq.* (1), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists\, [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$ *in the* δ*-layer of* $\partial\Omega_{ij}$. *Suppose* $\{\mathbf{x}_i^{(i)}(t_{m-}), \mathbf{x}_j^{(j)}(t_{m+})\} = \mathbf{x}_m$ *holds, the vector fields* $\mathbf{F}_i^{(i)}(t)$ *and* $\mathbf{F}_j^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t, respectively.* $\|\mathrm{d}^{r+1}\mathbf{x}_\alpha^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$ $(\alpha \in \{i,j\})$ *and* $m \in \{p,q,n\}$.

(1) *The switching conditions on* $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \overleftarrow{\partial\Omega}_{ij}$ *for passable flows in the* δ*-layer of* $\partial\Omega_{ij}$ *are*

$$
\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}_i^{(i)}(t_{n\pm}) = 0, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}_j^{(j)}(t_{n\pm}) = 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}_i^{(i)}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}_i^{(i)}(t_{q+})] < 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}_j^{(j)}(t_{p+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}_j^{(j)}(t_{q-})] < 0
\end{aligned} \tag{38}
$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{ij}$ *and* $\mathbf{x}_q \in \overleftarrow{\partial\Omega}_{ij}$ *and* $\lambda \in \{p,q\} \neq n$.

(2) *The switching conditions on* $\partial\Omega_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ *for source and sink motions in the* δ*-layer of* $\partial\Omega_{ij}$ *are*

$$\begin{aligned}
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{n\pm})] = [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{n\pm})] = 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(i)}_{i}(t_{q+})] < 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(j)}_{j}(t_{q+})] < 0
\end{aligned} \tag{39}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \widetilde{\partial\Omega}_{ij}$ *and* $\mathbf{x}_q \in \widehat{\partial\Omega}_{ij}$.

(3) *The onset and disappearance conditions on* $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widetilde{\partial\Omega}_{ij}$ *for the sliding motion in the* δ*-layer of* $\partial\Omega_{ij}$ *are*

$$\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\beta}(t_{n-}) = 0, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\alpha}(t_{n-}) \neq 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\alpha}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\alpha}(t_{q-})] > 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\alpha}(t_{p-})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\beta}(t_{q-})] < 0
\end{aligned} \tag{40}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ *and* $\mathbf{x}_q \in \widetilde{\partial\Omega}_{ij}$.

(4) *The onset and disappearance conditions on* $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{\alpha\beta} \cup \Gamma_{ij} \cup \widehat{\partial\Omega}_{ij}$ *for source motion in the* δ*-layer of* $\partial\Omega_{ij}$ *are*

$$\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\alpha}(t_{n+}) = 0, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\beta}(t_{n+}) \neq 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\beta}(t_{p+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\alpha)}_{\alpha}(t_{q+})] < 0,\\
&[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\beta}(t_{p+})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \bullet \mathbf{F}^{(\beta)}_{\beta}(t_{q+})] > 0
\end{aligned} \tag{41}$$

for $\mathbf{x}_n \in \Gamma_{ij}$, $\mathbf{x}_p \in \overrightarrow{\partial\Omega}_{\alpha\beta}$ *and* $\mathbf{x}_q \in \widehat{\partial\Omega}_{ij}$.

Proof. Using Definitions 1, 9 and 11, this theorem is proved directly. □

3. Conclusions

In this letter, the real and imaginary flows for non-smooth dynamical systems are introduced. The theory for real flows in Ref. [1] can be applied to the imaginary flows. The δ-layer of the separation boundary is also introduced. The onset, existence and disappearance of the sink and source flows in the δ-layer are discussed, and the switching between the two semi-passable flows and the switching between the sink and source flows are investigated as well. Finally, the necessary and sufficient conditions for the onset, disappearance and switching are presented. These conditions can be very easily applied to non-smooth dynamical systems in engineering, such as friction-induced vibrations, control systems with periodical excitations.

References

[1] A.C.J. Luo, A theory for non-smooth dynamical systems on the connectable domains, *Communications in Nonlinear Science and Numerical Simulation* 10 (2005) 1–55.